An advantage of this method is its flexibility in the sense that the assumptions for which it can be used are not very burdensome. Nevertheless, it should be noted that since the method is not based on an explicit probability model, its properties are not completely defined. Assuming also the fact that in practice all the subsequent processing of nonstationary random processes of the form (1) is based on the results of an estimate of a time-varying mean, we carried out a numerical experiment with the algorithm described by (6). The results showed that the region in which this algorithm can be used with a reasonable degree of accuracy corresponds to the region in which it can be used in the automated data-processing system. In Fig. 2 we present an estimate of the varying mean value of a nonstationary process obtained using this method.

All the algorithms considered form the basis of the software of an actual automated system for processing the data of a thermal experiment.

Notation

$\{x(\tau)\}$, a nonstationary random process; $\{y(\tau)\}$, an unknown random process (in a special case, a stationary random process); $A(\tau)$, a deterministic process; τ, time; $M[y(\tau)]$, mean value of the random process $\{y(\tau)\}$; $D[y(\tau)]=\sigma^2$, variance of the random process $\{y(\tau)\}$; R_i, difference between the recorded and predicted values; c, relative accuracy; $\hat{\mu}$, estimate of the mean value of the random process; C_s, a weighting coefficient; and T, observation period.

LITERATURE CITED

1. B. M. Pankratov, Yu. V. Polezhaev, and A. K. Rud'ko, Interaction between Materials and Gas Flows [in Russian], Mashinostroenie, Moscow (1976).
2. T. Anderson, Statistical Analysis of Time Series, Wiley (1971).
3. J. Bendat and A. Piersol, Random Data Analysis and Measurement Procedures, Wiley (1971).

THE PROBLEM OF PLANNING THERMAL MEASUREMENTS

S. A. Budnik

UDC 536.24.083

The solution of the problem of planning thermal measurements is considered. Cubic splines are used for a mathematical description of the relations investigated.

Optimum planning of measurements is being increasingly introduced into experimental research at the present time. A fairly full review of papers devoted to this question is contained in [1, 2]. A large number of papers are devoted to the theory and general methods of solution, but a much smaller number consider the practical application of these methods for planning physical, and in particular, thermal (laboratory and natural) experiments [3-5]. This is largely due to the complexity of the mathematical description of nonstationary heat and mass transfer processes which are encountered in practice, and also due to the rigid limitations imposed on the number of repeated experiments, due to the considerable material costs in preparing and carrying them out.

Methods of planning, which are becoming more and more widely used, are based on the assumption that the process is linear with respect to the parameters of the mathematical model of the process being investigated or on the possibility of linearizing it [3, 6, 7]. In addition, a number of fairly rigid constraints are imposed on the chosen model and on the required optimum plan, such as the requirement that the points of the plan should be orthogonal, that these points should be symmetrical, etc. [6]. In the case of several monitored variables, when the model is nonlinear, or when the conditions under which the experiment is carried out change with time, these methods are inapplicable and it is necessary to employ sequential planning.

Below we consider the problem of planning measurements when carrying out a thermal experiment. The purpose of the experiment is to determine the distribution of the parameter of nonstationary heat exchange $P(x, \tau)$ on the surface of a solid as a function of the coordinate x and the time τ. These parameters may be the pressure, convective and radiation thermal flows, the surface temperature, removal of mass, etc.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 39, No. 2, pp. 225-230, August, 1980. Original article submitted January 2, 1980.

0022-0841/80/3902-0840$07.50

The problem will be formulated as follows: it is required to determine the number of measurements (the number of pickups n and number of interrogations of each pickup m_k) and the measurement plan (the arrangement of the measurements in space $\varepsilon_x = \{x_k\}$, $k = 0, 1, \ldots, n$, and time $\varepsilon_\tau^k = \{\tau_l\}$, $l = 0, 1, \ldots, m_k$) so that when these measurements are carried out and processed the investigated relationship $P(x, \tau)$ $(x_a \leqslant x \leqslant x_b,\ \tau_a \leqslant \tau \leqslant \tau_b)$ will be obtained with a specified accuracy at minimum cost (the minimum number of measurements $M = \sum_{k=1}^{n} m_k$) or, if the number of measurements is restricted, with maximum accuracy for a specified number of measurements.

We make the following assumptions: the parameter being investigated is measured directly, the results of the measurements $Y_{k,l}$ at points of the plan x_k, τ_l, $k=0, 1, \ldots, n$, $l=0, 1, \ldots, m_k$, contain measurement errors, and for the average value of $Y_{k,l}$ the following condition is satisfied:

$$E(Y/\bar{z}) = P(\bar{z}). \tag{1}$$

Here $\bar{z}^T = \|x, \ \tau\|$ is the vector of the monitored variables, and E is the averaging operator. The measurement errors are in the form of random noise with zero expectation and variance $\sigma^2(\bar{z})$. The errors in determining x and τ are ignored, and information on the nature of the relationship $P(\bar{z})$ and the measurement errors are specified a priori.

We will use cubic splines for a mathematical description of the relationship P(z). This type of function is a fairly universal and accurate means of representing the relationships considered, and it is possible to compile relatively simple algorithms to construct them [8, 9].

The method of processing the measurement results is determined by the choice of the mathematical description of $P(\bar{z})$ and consists in constructing cubic splines from the results of the measurements at points of the optimum plan. Since results of the measurements contain random errors, we chose smoothing splines.

The function $S(x) \in C^2(x_a, x_b)$ that minimizes the integral

$$\int_{x_a}^{x_b} [S''(x)]^2 dx \tag{2}$$

will be called a cubic smoothing spline when the following inequality is satisfied:

$$\sum_{k=0}^{n} \left[\frac{S(x_k) - Y_k}{\delta Y_k}\right]^2 \leqslant \alpha, \tag{3}$$

where Y_k are the values of the smoothed function at nodal points of the spline x_k; $\alpha \geqslant 0$ represents the degree of smoothing, and S(x) in each section $[x_{k-1}, x_k]$ of the network $\Delta_k : x_a = x_0 < x_1 < \ldots < x_n = x_b$ is a cubic polynomial of the form

$$S(x) = S_k(x) = \sum_{i=0}^{3} a_{k,i}(x_k - x)^i, \quad k = 0, 1, \ldots, n, \tag{4}$$

and satisfies the boundary condition

$$S''(x_0) = S''(x_n) = 0. \tag{5}$$

When $\alpha = 0$ the smoothing spline becomes an interpolating spline. The value of δY_k is related to the variance of the measurement error by the equation

$$\delta Y_k = \sqrt{\sigma_k^2}, \quad k = 0, 1, \ldots, n. \tag{6}$$

In the majority of thermal experiments the change in the plan ϵ_x during the experiment is a technically complex problem, and is often unsolvable in practice. It is therefore advisable to divide the planning problem into two for the variables x and τ respectively, i.e., we first construct a plan ϵ_x that is optimum for the whole range of variation of τ, and then determine ϵ_τ^k.

We will choose as the criterion of optimality of the plan ϵ_x the minimum of the functional

$$F_0 = \sum_{i=0}^{L} \sum_{j=0}^{N} [S(x_j, x_k, \tau_i) - P(x_j, \tau_i)]^2 \to \min_{x_k, n}, \quad k = 0, 1, \ldots, n, \tag{7}$$

the values of which represent the sum of the squares of the deviations of the spline $S(x, x_k)$, which approximates the values $Y_{k,i}$ with respect to the relationship P(x) at each instant of time τ_i. The values of $Y_{k,i}$ are formed on the basis of a table of the specified relationship $P(x_j, \tau_i)$, $\{[x_0, x_N][\tau_0, \tau_L]\}$, on which random noise with variance $\sigma^2(x_j, \tau_i)$ is superimposed.

The condition for choosing the plan of interrogation of the k-th pickup ϵ_τ^k has the form

$$F_1 = \sum_{i=0}^{L} [S(\tau_i, \tau_l) - P(\tau_i, x_k)]^2 \to \min_{\tau_l, m_k}, \quad l = 0, 1, \ldots, m_k, \tag{8}$$

where $S(\tau_i, \tau_l)$ are the values of the corresponding spline, and τ_l are its nodal points.

If, in the conditions under which the experiment is carried out, it is necessary to construct an overall plan ϵ_τ, optimum for all the pickups, we obtain

$$F_2 = \sum_{k=0}^{n} \sum_{i=0}^{L} [S(\tau_i, \tau_l, x_k) - P(\tau_i, x_k)]^2 \to \min_{\tau_l, m}, \quad l = 0, 1, \ldots, m. \tag{9}$$

Here $S(\tau_i, \tau_l, x_k)$ are the values of the smoothing spline which approximates Y_l at each point x_k. For measurements that are continuous in time we will only solve problem (7).

Hence, in the formulation of the problem considered the points of the optimum plan are the normal points of the corresponding splines, and the solution reduces to constructing optimum splines in the sense of (7), (8), or (9). The procedure for solving problem (7) is as follows.

1. We specify $n \geqslant 3$; as a rule it is convenient to begin with $n = 3$.

2. We solve problem (7) for the chosen n; the method of conjugate gradients [10] is used to minimize the quadratic functional. The iterational process with respect to x_k, $k = 0, 1, \ldots, n$, is terminated when the following condition is satisfied:

$$F_0^l(\varepsilon_x^l, n) \leqslant \delta, \tag{10}$$

where $\delta > 0$ is a previously specified number chosen from the requirements regarding the accuracy of the specific experiment, or the condition

$$\max_{x_k^l} \left| \frac{x_k^l - x_k^{l+1}}{x_k^l} \right| \leqslant \delta_1, \quad k = 0, 1, \ldots, n, \tag{11}$$

where $\delta_l > 0$ is a previously specified small number which is chosen depending on the accuracy with which the probes are set up with respect to the coordinate x.

If condition (10) is satisfied we proceed to step 4, and if not condition (11) is checked. If condition (11) is satisfied we transfer to step 3, otherwise the solution of problem (7) is continued for the chosen n.

3. In the plan ϵ_x one point $n = n + 1$ is added and we then proceed to step 2.

4. End of the problem.

We can set up a procedure for solving problems (8) and (9) in a similar manner. To construct smoothing splines we used the modified algorithm described in [9].

Depending on the amount and nature of the a priori information of $P(\bar{z})$ and $\sigma^2(\bar{z})$, two approaches to solving the problem are possible. For fairly complete information, e.g., when these relationships are known analytically or in the form of tables from the results of a numerical simulation or experiments, while the aim of the planned measurements is to refine the relationship P(z), a static method of planning is used. The lack of information requires the use of sequential planning, which consists of a series of stages: planning of the carrying out of the measurements, processing of the results, planning, etc. At each stage of the planning the problem is solved using the results of the preceding stages as a priori information.

To estimate the efficiency of the algorithm we solved a number of model examples. The results of the solution of two of these are presented below. In the first case we solved problem (7) assuming that the measurements were carried out continuously in time and without errors. Information on the relation $P(x, \tau)$ investigated was specified in the form of a table of $P(x_j, \tau_i)$ in steps of $\Delta x = 0.1$ and $\Delta\tau = 1$ sec. Figures 1a and b show $P(x, \tau)$ with a step of $\Delta\tau = 2$ sec, which enables the nature of the curves to be estimated as well as the values for $n = 8$ and $\tau = 12$, 20, 30, and 38 sec.

Table 1 shows the coordinates of the optimum plans with a number of points n = 3-8, and values of the parameters representing the accuracy of the reconstruction $P(x, \tau)$ from the results of measurements at the chosen points

$$\bar{F} = \frac{F_0}{\sum_{i=0}^{L} \sum_{j=0}^{N} [P(x_j, \tau_i)]^2}; \tag{12}$$

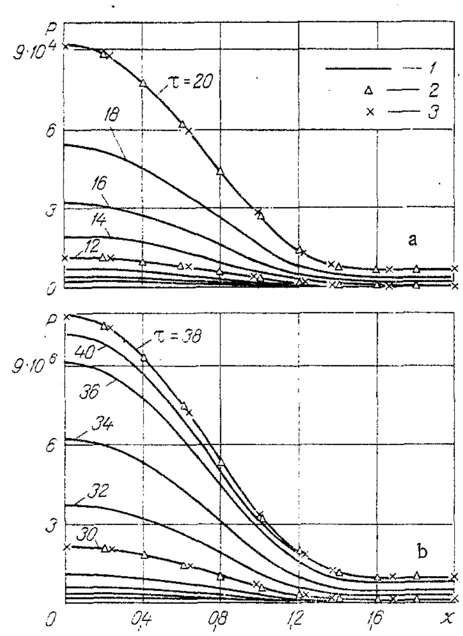

Fig. 1. Dependence of the parameter P on the dimensionless coordinate x and the time τ (sec) for τ = 1-20 sec (a) and τ = 22-40 sec (b): 1) the relationship $P(x, \tau)$; 2) values reconstructed using the spline with n = 8, for τ = 12, 20, 30, and 38 sec; 3) results of measurements.

TABLE 1. Coordinate of the Points of the Optimum Plans and Characteristics of the Accuracy with Which the Relation $P(x, \tau)$ is Reconstructed

n	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\bar{F}_p$	$\bar{F}$	Q
3	0,00	0,89	2,00						$2{,}6\cdot10^{-2}$	$2{,}4\cdot10^{-2}$	0,152
4	0,00	0,28	1,04	2,00					$3{,}6\cdot10^{-3}$	$2{,}5\cdot10^{-4}$	0,025
5	0,00	0,30	1,05	1,54	2,00				$2{,}5\cdot10^{-4}$	$1{,}4\cdot10^{-4}$	0,020
6	0,00	0,40	0,80	1,20	1,60	2,00			$1{,}3\cdot10^{-4}$	$1{,}1\cdot10^{-4}$	0,017
7	0,00	0,23	0,67	1,06	1,32	1,67	2,00		$1{,}1\cdot10^{-4}$	$6{,}3\cdot10^{-5}$	0,015
8	0,00	0,22	0,63	0,98	1,21	1,36	1,67	2,00	$7{,}0\cdot10^{-5}$	$5{,}8\cdot10^{-5}$	0,012

TABLE 2. Coordinates of the Points of the Optimum Plans and the Characteristics of the Accuracy with Which the Relationship $P(x)$ is Reconstructed

n	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{F}_p$	$\bar{F}$	Q
3	0,00	0,44	1,00				$1{,}2\cdot10^{-2}$	$1{,}1\cdot10^{-2}$	0,108
4	0,00	0,47	0,68	1,00			$1{,}3\cdot10^{-2}$	$1{,}0\cdot10^{-2}$	0,071
5	0,00	0,28	0,52	0,72	1,00		$7{,}8\cdot10^{-3}$	$1{,}4\cdot10^{-3}$	0,047
6	0,00	0,14	0,39	0,57	0,89	1,00	$5{,}3\cdot10^{-3}$	$9{,}9\cdot10^{-4}$	0,032

$\bar{F}_p$ is a quantity similar to $\bar{F}$, calculated for a uniform plan ϵ_x ;

$$Q = \frac{\max\limits_{j,i}|S(x_j,\ x_h,\ \tau_i) - P(x_j,\ \tau_i)|}{\max\limits_{j,i}|P(x_j,\ \tau_i)|}. \tag{13}$$

These results show that the chosen optimum plan with n = 8 gives high accuracy in the reconstruction of $P(x, \tau)$ over the whole range of variation of the monitored variables. The maximum relative error Q = 1.2%.

The error in reconstructing $P(x, \tau)$ decreases as n increases. However, beginning at a certain n, any further increase in the number of points ceases to have any effect on the error in practice. This property of the relationships $\bar{F}(n)$ and $Q(n)$ must be taken into account when determining the value of δ in (10).

Comparing the values of $\bar{F}$ and $\bar{F}_p$ shown in Tables 1 and 2, we note that for small values of n the accuracy with which $P(x, \tau)$ is reconstructed depends to a considerable extent on the manner in which it varies. For large n the optimum plans differ only slightly from uniform plans.

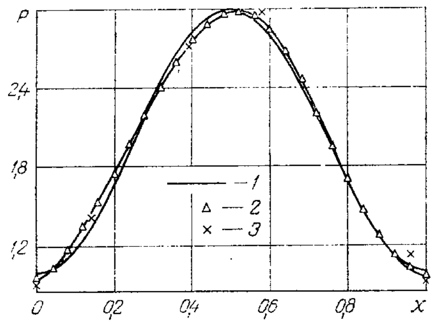

Fig. 2. Curves of P as a function of the dimensional coordinate x: 1) the relationship P(x); 2) the relationship reconstructed using a spline with n = 6, 3) measurement results.

We also solve the problem of planning for the relationship

$$P(x) = 2 - \cos(2\pi x) \quad (14)$$

in the interval [0, 1] when measurement errors are present in the form of random noise with a normal distribution N(0; 0; 0.1). The relation P(x) was specified in the form of a table $P(x_j)$, j = 0, 1, ..., N, with a step $\Delta x = 0.04$. The variances of the measurement errors $\sigma^2 = \text{const} = 0.01$.

Figure 2 shows curves of P(x) and a curve reconstructed from the results of measurements at points of the optimum plan n = 6. The results of the measurements were simulated by superimposing on the P(x) curve the component random noise with a normal distribution N(0; 0; 0.1).

Table 2 shows coordinates of the optimum plans for n = 3-6 and values of the parameters $\bar{F}(n)$, $\bar{F}_p(n)$, and Q(n). These results show that the conclusions drawn previously hold for the last example.

NOTATION

P, parameter being investigated; x, coordinate; τ, time (sec); n, m, number of points of the plan; ϵ_x, ϵ_τ, measurement plans; M, total number of measurements; $Y_{k,l}$, results of measurements; x_k, τ_k, coordinates of the points of the plans; σ^2, variance of the measurement errors; S, a cubic spline; δY_k, mean square deviation of the ordinate; α, a smoothing parameter; F, a functional; δ, a specified constant; $\bar{F}$, Q, characteristics of the reconstruction accuracy; $\bar{F}_p$, characteristics of the reconstruction accuracy for a uniform plan; and Δx, $\Delta\tau$, steps of the table.

LITERATURE CITED

1. Yu. P. Adler and Yu. V. Granovskii, "Review of practical work on the planning of experiments," Preprint No. 33, Moscow State Univ., Moscow (1972).
2. V. P. Shlykova, "Planning and automation of an experiment in scientific research," Index of Soviet and Foreign Literature 1973-1976, MÉI, Moscow (1976).
3. E. I. Taubman and Yu. I. Kalishevich, "The problem of planning thermal experiments," Inzh.-Fiz. Zh., 25, No. 2, 345-348 (1973).
4. A. I. Lyubarskii, "Mathematical planning of a heat and mass transfer experiment," Proc. of the Fifth All-Union Conference on Heat and Mass Transfer, Minsk (1976), pp. 110-113.
5. V. G. Bogdanov and S. V. Epifanov, "Optimum planning of experiments to determine the boundary conditions of heat transfer," Proc. of the Fifth All-Union Conference on Heat and Mass Transfer, Minsk (1976), Naukova Dumka, Kiev (1976), pp. 120-125.
6. V. V. Fedorov, Theory of the Optimum Experiment [in Russian], Nauka, Moscow (1971).
7. V. K. Kaishev, "Planning an experiment in problems of constructing spline-regression models," Tr. MÉI, No. 300, 105-108 (1976).
8. S. B. Stechkin and Yu. N. Subbotin, Splines in Computational Mathematics [in Russian], Nauka, Moscow (1976).
9. C. H. Reinsch, "Smoothing by spline functions," Num. Math., 10, No. 3, 177-183 (1967).
10. B. N. Pshenichnyi and Yu. M. Danilin, Numerical Methods in Extremal Problems [in Russian], Nauka, Moscow (1975).